Ecole Nationale Polytechnique

Preparatory cycle

Exercise series n ° 1 - Semester 1

2025-2026

Introduction - Revision - Mathematical logic - Theory of sets

$\mathbb{N}$ denotes the set of natural numbers.
$\mathbb{Z}$ denotes the set of integers.
$\mathbb{Q}$ denotes the set of rational numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathbb{C}$ denotes the set of complex numbers.

The sign $\sum$ denotes the sum and $\prod$ denotes the product.

$$\sum_{k=1}^{n} k = 1+2+3+...+n = \frac{n(n+1)}{2} \qquad \prod_{k=1}^{n} k = 1\times 2\times 3\times ...\times n = n!$$

Example 1 :

$\forall n \in \mathbb{N}$, let : $S_1 = \sum\limits_{k=0}^{n} k$ and $S_2 = \sum\limits_{k=0}^{n} k^2$

1. Demonstrate that : $\forall n \in \mathbb{N}, S_1 = \dfrac{n(n+1)}{2}$.
2. Using the inequality
$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n+1}((k-1)+1)^3,$$
and by expanding the second member, find the value of the sum S_2.

Example 2 :

Calculate the following product : $\prod\limits_{k=2}^{n}(1-\dfrac{1}{k^2})$.

Exercise 1 : The rain and the umbrella

The following two statements are given and assumed to be true :
"If it rains, Ali takes an umbrella".
"Houda never takes an umbrella if it's not raining and always takes one when it is".
The following three logical propositions are introduced : P , Q and R :
P : It's raining. Q : Ali has an umbrella. R : Houda has an umbrella.

1. Write the statement using these propositions and the logical connectors : $\Rightarrow$ and $\Leftrightarrow$.
2. What can we deduce from these statements in the various situations below :
   a. Ali is walking with an umbrella.
   b. Ali is walking without an umbrella.
   c. Houda is walking with an umbrella.
   d. Houda walks without an umbrella.
   e. It's not raining.
   f. It's raining.
   Justify your answers carefully.

Exercise 2 :
a.Let $a, b, c \in \mathbb{R}$. give the negation of the following propositions :

1. $a \leq -2$ or $a \geq 3$;
2. $a \leq 5$ and $a > -1$;
3. $a \leq 5$ or $3 > c$.

b.Let x, y, a, b be real numbers. Say whether the following propositions are true or false. Justify.

1. $(x \leq 2) \Rightarrow (x^2 \leq 4)$
2. $0 \leq x \leq y$ et $a \leq b \Rightarrow (xa \leq yb)$
3. $(xy \neq 0) \wedge (x \leq y) \Rightarrow \left(\frac{1}{x} \geq \frac{1}{y}\right)$

c. Complete the dotted lines with the appropriate logical connector : $\Rightarrow, \Leftarrow, \Leftrightarrow$.

1. $x \in \mathbb{R},\ \ x^2 = 4 \ldots x = 2$;
2. $z \in \mathbb{C},\ \ z = \bar{z} \ldots z \in \mathbb{R}$;
3. $x \in \mathbb{R},\ \ x = \pi \ldots e^{2ix} = 1$.

Exercise 3 :
Deny the following assertions :

1. Every right-angled triangle has a right angle ;
2. In all stables, all horses are black ;
3. For any integer x, there exists an integer y such that, for any integer z, the relation $z < x$ implies the relation $z < x+1$;
4. $\forall \varepsilon > 0, \exists \alpha > 0 \ \ s.t. \ \ (|x - 7/5| < \alpha) \Rightarrow |5x - 7| < \varepsilon)$.

Exercise 4 :

1. Show by two reasonings that : $x \notin \mathbb{Q} \Rightarrow 1 + x \notin \mathbb{Q}$.
2. Show by two types of reasoning that if $a^2 + 9 = 2^n$ then a is odd.
3. Show by recurrence the property : $\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx$.
4. Show using the contrapositive that if the integer $(n^2 - 1)$ is not divisible by 8 then the integer n is even.
5. Show that if m and n are odd integers then $m^2 + n^2$ is even but not divisible by 4.

Exercise 6 :
Let E be a non-empty set. $P(E)$ denotes the set of parts of E.

1. Show that the following equivalence is true : $\forall A, B \in P(E) : (A \cap B = A \cup B) \Rightarrow A = B$.
2. Show that the following implications are true :
   a.$\forall A, B \in P(E)$, $(A \cap B = A \cap C) \wedge (A \cup B = A \cup C) \Rightarrow B = C$.
   b.$(A \cap B = A \cap C) \Rightarrow (A \cap \bar{B} = A \cap \bar{C})$.
   We will use direct reasoning, as well as contrapositive reasoning.

Exercise 7 : Justify the following equalities :

1. $\forall A, B \subset E;\ \ (A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$: distributivity of the $\cap$ law with respect to the $\triangle$ law
2. $\forall A, B \subset E; A \triangle B = \bar{A} \triangle \bar{B}$:Here $\bar{A}$ means $A - B$ and $\bar{B}$ means $B - A$.