National Polytechnic School
Preparatory cycle

Exercise series n ° 4 - Semester 1
2025-2026
Algebraic structures

Exercise 1 :

1. In $\mathbb{R}$, we define the closed operation $\star$ by : $\forall x, y \in \mathbb{R}$: $\quad x \star y = xy + (x^2-1)(y^2-1)$

(a). Is $\star$ commutative ? justify.

(b). Verify that $\mathbb{R}$ has an identity element for $\star$.

(c). Calculate $(2 \star 2) \star 3$ and $2 \star (2 \star 3)$.

(d). Calculate the symetric(s) of 2 for the operation $\star$.

2. Let $\star$ a closed operation in $\mathbb{R}_+^*$ defined by :

$$\forall x, y \in \mathbb{R}_+^* : \quad x \star y = \sqrt{x^2+y^2}$$

(a). Sow that no element in $\mathbb{R}_+^*$ has a symmetric for the operation $\star$ in $\mathbb{R}_+^*$.

Exercise 2 :

We define an internal composition law $\star$ on $\mathbb{R}$ by $\forall (a,b) \in \mathbb{R}^2$, $a \star b = \ln(e^a + e^b)$.
What are its properties (commutativity, associativity, e and a^{-1} ?

Exercice 3 :

Show that the following laws equip the whole G indicated with a group structure, and specify if it is Abelian :

1. $x \star y = \dfrac{x+y}{1+xy}$ on $G =]-1,1[$.
2. $(x,y) \star (x',y') = (x+x', y\exp(x') + y'\exp(-x))$ on $G = \mathbb{R}^2$.

Exercise 4 :

Let $(E,*)$ be a structured set. An element a of E is said to be simplified for the left of $*$ law if

$$\forall (x,x') \in E^2 : a * x = a * x' \Rightarrow x = x'.$$

An element $a \in E$ is said to be simplified for the tight of $*$ law if

$$\forall (x,x') \in E^2 : x * a = x' * a \Rightarrow x = x'.$$

Show that if $(G,*)$ is a group then every element of G can be simplified on the left and right for the $*$ law.

Exercise 5 :

Let $(G,*)$ be a group and $a \in G$. We define an internal composition law $\dagger$ on G by $x \dagger y = x * a * y$.

1. Show that $(G, \dagger)$ is a group.
2. Let H a subgroup of $(G,*)$ and $K = a' * H = \{a' * x / x \in H\}$.
   Show that K is a subgroup of $(G,*)$.
3. Show that $f : x \to x * a'$ is an isomorphism of $(G,*)$ towards $(G,\dagger)$.

Exercice 6 :

Let $E = f : \mathbb{C} \to \mathbb{C}$, homographic functions, ie $\exists\ (a, b, c, d) \in \mathbb{R}^4 : f(z) = \dfrac{az+b}{cz+d}$, with $ad - bc$ non zero.
We say that (a, b, c, d) are parameters of f, said transformation of Mobius f.
Let $\triangle$ the application of $E \longrightarrow \mathbb{R} - \{0\}$ such that $\triangle(f) = ad - bc$.
Let G the subset of E such that $G = \triangle^{-1}(\{-1, 1\})$.

1. Justify that every function f of E is injective.
2. Prove that : $\forall f \in E, \exists\ k \in \mathbb{R}^*$ and $\exists\ g \in G$, such that $f = kg$.
3. Determine, depending on (a, b, c, d) of $f \in G$, the parameters $(\alpha, \beta, \gamma, \delta)$ of symmetric F within the group (G, o).
4. Is (G, o) an abelian group ? Justify.
5. Determine, for all $(f, g) \in E^2, \Delta(fog)$ depending on $\Delta(f)$ and $\Delta(g)$.
   (a) What does the application Δ represent for the groups (G, o) and $(\mathbb{R} - \{0\}, \times)$?
   (b) Determine $\Delta^{-1}(\{1\})$ and explain the meaning of this set.
6. Let $(f, g) \in G^2$.
   (a) What are the values of $(\Delta(f), \Delta(g))$ when $fog \in \Delta^{-1}(\{-1\})$.
   (b) Same question with $fog \in \Delta^{-1}(\{1\})$.

Exercise 8 :

We define on $\mathbb{Z}^2$ two closed operations noted $+$ et $*$ by :
$(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) * (c, d) = (ac, ad + bc)$.

1. Show that $(\mathbb{Z}^2, +, *)$ is a commutative ring.
2. Show that $A = (a, 0)/a \in \mathbb{Z}$ is a sub ring of $(\mathbb{Z}^2, +, *)$

Exercise 9 :

Let the two operations in $\mathbb{R}^2$ definid by :

$$(x, y) + (x', y') = (x + x', y + y') \qquad \text{and } (x, y) \times (x', y') = (xx', xy' + x'y)$$

1. Show that $(\mathbb{R}^2, +, \times)$ is a commutative ring with one ;
2. Is this ring an integral domain ? a field ? Justify ;
3. Let the application defined by :

$$\begin{array}{rccc} \varphi : & (\mathbb{R}^2, +) & \to & (\mathbb{R}, +) \\ & (x, y) & \mapsto & x - y \end{array}$$

(a). Show that φ is a morphism of groups ;
(b). Determine $\ker\varphi$ and $Im\varphi$. Is φ an isomorphisme ? Justify.

Exercise 10 :

We notice $\mathbb{Z}$ the following set of reals :

$$\mathbb{Z}[\sqrt{2}] = \{m + n\sqrt{2}, m, n \in \mathbb{Z}\}$$

1. Show that $\mathbb{Z}[\sqrt{2}]$, provided with the addition and multiplication of real numbers, is a subring of $\mathbb{R}$.
2. Let $\phi(m + n\sqrt{2}) = m - n\sqrt{2}$.
   Show that ϕ is an automorphism of the ring $(\mathbb{Z}[\sqrt{2}], +, \times)$ (it is a bijection, and a morphism for each of the two laws).
3. For all $x \in \mathbb{Z}[\sqrt{2}]$, let $N(x) = x\phi(x)$. Show that N is an application of $\mathbb{Z}[\sqrt{2}]$ in $\mathbb{Z}$,which is a morphism for multiplication.

Exercise 11 :

1. Show that the set $A = \mathbb{R}$ provided with each of the two laws : $x * y = xy - x - y + 2$ and $xTy = x + y - 1$ is a magma.

2. Justify that $(A, T, *)$ is a commutative ring. Is it an integral domain ?
3. Express nx where n is a natural number ≥ 2 and $x \in A$, with notations :
   $2x$ the element xTx, $3x$ the element $(xTx)Tx = xT(xTx)$...

Exercise 12 :

Let $\mathbb{K}$ the set of complex numbers which are written as $z = r + is$ where $r \in \mathbb{Q}$ and $s \in \mathbb{Q}$.

1. Show that $(\mathbb{K}, +)$ is a commutative subgroup.
2. Show that $(\mathbb{K}^*, .)$ is a commutative subgroup.
3. Deduce that $(\mathbb{K}, +, .)$ is a commutative field.