Chapter III: The ring of polynomials and the field of rational fractions

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Teaching content

Introduction

- Polynomials are very simple objects but with extremely rich properties. You already know how to solve the equations of degree 2 : $aX^2 + bX + c = 0$. Do you know that solving equations of degree 3, $aX^3 + bX^2 + cX + d = 0$, was the subject of fierce struggles in Italy of the xvi^e century ?
- A fundamental theorem of algebra : « Every polynomial of degree n has n complex roots. »

Here, $(\mathbb{K}, +, \times)$ is a commutative field witch can represent :

- The field $\mathbb{Q}$,
- The field $\mathbb{R}$,
- The field $\mathbb{C}$,

Definitions, notations

Polynomial function, formal polynomial

- A polynomial function with coefficient in $\mathbb{K}$ is :

$$P(X)=\sum_{i=0}^{i=n} a_i X^i = a_0X^0+a_1X^1+\cdots+a_nX^n$$

- A formal polynomial with coefficient in $\mathbb{K}$ is a sequence $(a_n)_{n\in\mathbb{N}}$ on $\mathbb{K}$, where $\exists N\in\mathbb{N} : \forall n\in\mathbb{N} \ \ (n>N \Longrightarrow a_n=0)$.

With $n\in\mathbb{N}$ and $a_0, a_1, \ldots, a_n\in\mathbb{K}$. The set of polynomials is noted $\mathbb{K}[X]$.

$\rightsquigarrow$ $X^3-5X+\frac{3}{4}$ is a polynomial of degree 3

$\rightsquigarrow$ X^n+1 is a polynomial of degree n.

$\rightsquigarrow$ 2 is constant polynomial of degree 0.

Basics notations

Basics notations

- The a_i are coefficients of the polynomial.
- If all the coefficients a_i are zero, P is called the zero polynomial, it is denoted 0.
- We call the degree of P the largest integer i such that $a_i \neq 0$; we denote it $\deg P$. For the degree of the zero polynomial we set by convention $\deg(0) = -\infty$.
- A polynomial of the form $P = a_0$ with $a_0 \in \mathbb{K}$ is called a constant polynomial. If $a_0 \neq 0$, its degree is 0.
- If P is in degree n, then (a_n) is the **dominant** coefficient. If $(a_n) = 1$. then P is unitary.

Basics operations

Equality

Let $P = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ and
$Q = b_nX^n + b_{n-1}X^{n-1} + \cdots + b_1X + b_0$ two polynomial on $\mathbb{K}$.

$$P = Q \quad \Longleftrightarrow \quad \forall i \quad a_i = b_i$$

we say that P are Q equal.

Addition

Let $P = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ and
$Q = b_nX^n + b_{n-1}X^{n-1} + \cdots + b_1X + b_0$.
We define :

$$P+Q = (a_n+b_n)X^n+(a_{n-1}+b_{n-1})X^{n-1}+\cdots+(a_1+b_1)X+(a_0+b_0)$$

Basics operations

Multiplication

Let $P = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ and $Q = b_mX^m + b_{m-1}X^{m-1} + \cdots + b_1X + b_0$. We define

$$P \times Q = c_rX^r + c_{r-1}X^{r-1} + \cdots + c_1X + c_0$$

$$\text{with } r = n + m \text{ and } c_k = \sum_{i+j=k} a_ib_j \text{ pour } k \in \{0, \ldots, r\}.$$

Multiplication by scalar

If $\lambda \in \mathbb{K}$ then $\lambda \cdot P$ is the polynomial with the i-the coefficient is λa_i.

Basics operations-example

Basics operations-example

Consider the two polynomials $P = aX^3 + bX^2 + cX + d$ and $Q = \alpha X^2 + \beta X + \gamma$.

- $P + Q = aX^3 + (b + \alpha)X^2 + (c + \beta)X + (d + \gamma)$;
- $P \times Q = (a\alpha)X^5 + (a\beta + b\alpha)X^4 + (a\gamma + b\beta + c\alpha)X^3 + (b\gamma + c\beta + d\alpha)X^2 + (c\gamma + d\beta)X + d\gamma$.
- $P = Q$ if and only of $a = 0$, $b = \alpha$, $c = \beta$ and $d = \gamma$.
- The multiplication by scalar par $\lambda \cdot P$ is equivalent to multiply the constant polynomial λ by the polynomial P.

Basics Operations

Proposition

If $P, Q, R \in \mathbb{K}[X]$ then

- $0 + P = P, \quad P + Q = Q + P,$
  $(P + Q) + R = P + (Q + R);$
- $1 \cdot P = P, \quad P \times Q = Q \times P, \quad (P \times Q) \times R = P \times (Q \times R);$
- $P \times (Q + R) = P \times Q + P \times R.$
- $deg(P \times Q) = degP + degQ$
- $deg(P + Q) \leqslant \max(degP, degQ)$

We note $\mathbb{R}_n[X] = \{P \in \mathbb{R}[X] \mid \deg P \leqslant n\}$. If $P, Q \in \mathbb{R}_n[X]$ then $P + Q \in \mathbb{R}_n[X]$.

Basics Operations

Division

Let $A, B \in \mathbb{K}[X]$, we say that B **divide** A if the exist $Q \in \mathbb{K}[X]$ such that $A = BQ$. We note $B|A$.

Others evident property as $A|A$, $1|A$ and $A|0$ we have : for $A, B, C \in \mathbb{K}[X]$.

1. if $A|B$ et $B|A$, then there exist $\lambda \in \mathbb{K}^*$ such that $A = \lambda B$.
2. If $A|B$ and $B|C$ then $A|C$.
3. If $C|A$ and $C|B$ then $C|(AU + BV)$, for all $U, V \in \mathbb{K}[X]$.

Basics Operations

Euclidean Division theorem

Let $A, B \in \mathbb{K}[X]$, where $B \neq 0$, there exist a unique polynomial Q and there exist a unique polynomial R such that :

$$A = BQ + R \quad \text{and} \quad \deg R < \deg B$$

- Q is the **quotient** and R is the **reminder** of the **euclidean division** of A by B.
- The condition $\deg R < \deg B$ means that $R = 0$ or $0 \leqslant \deg R < \deg B$.
- $R = 0$ if and only if $B|A$.

Euclidean Division example

$$A(X) = X^3+X^2-1, B(X) = X-1, Q(X) = X^2+2X+2, R(X) = 1$$

$$\begin{array}{rrrr|lll}
x^3 & +x^2 & & -1 & x & -1 & \\ \cline{5-7}
-x^3 & +x^2 & & & x^2 & +2x & +2 \\ \cline{1-2}
 & 2x^2 & & & & & \\
 & -2x^2 & +2x & & & & \\ \cline{2-3}
 & & 2x & & & & \\
 & & -2x & +2 & & & \\ \cline{3-4}
 & & & +1 & & &
\end{array}$$

Euclidean Division example

$A(X) = X^n - 1, B(X) = X - 1, Q(X) = X^{n-1} + X^{n-2} + \cdots + 1, R(X) = 0$

$$
\begin{array}{cccc|c}
x^n & & & -1 & x \quad -1 \\
\cline{5-5}
-x^n & +x^{n-1} & & & x^{n-1} \; +x^{n-2} \; \cdots + 1 \\
\cline{1-2}
 & x^{n-1} & & -1 & \\
 & -x^{n-1} & +x^{n-2} & & \\
\cline{2-3}
 & & x^{n-2} & -1 & \\
 & \ddots & & & \\
 & & & x-1 & \\
 & & & -x+1 & \\
\cline{3-4}
 & & & 0 &
\end{array}
$$

GCD and LCD

Greatest common divisor

Let $A, B \in \mathbb{K}[X]$, where $A \neq 0$ or $B \neq 0$. There exists a unique unit polynomial of greatest degree which divides both A and B. This unique polynomial is called the **GCD** (greatest common divisor) of A and B which we note $gcd(A, B)$.

Least common divisor

Let $A, B \in \mathbb{K}[X]$ be non-zero polynomials, then there exists a unique unit polynomial M of smallest degree such as $A|M$ and $B|M$. This unique polynomial is called the **LCD** (least common divisor) of A and B which we note $lcd(A, B)$.

Euclide's algorithm

Euclide's algorithm

Let A and B two polynomials, $B \neq 0$. We calculate the successive Euclidean divisions.

$$\begin{array}{ll}
A = BQ_1 + R_1 & \deg R_1 < \deg B \\
B = R_1 Q_2 + R_2 & \deg R_2 < \deg R_1 \\
R_1 = R_2 Q_3 + R_3 & \deg R_3 < \deg R_2 \\
\vdots & \\
R_{k-2} = R_{k-1} Q_k + R_k & \deg R_k < \deg R_{k-1} \\
R_{k-1} = R_k Q_{k+1} &
\end{array}$$

The degree of the remainder decreases with each division. We stop the algorithm when the remainder is zero. The gcd is the last non-zero remainder R_k (unitary rendering).

Example

Euclide's algorithm application

The gcd of $A = X^5 + X^4 + 2X^3 + X^2 + X + 2$ and
$B = X^4 + 2X^3 + X^2 - 4$.
$X^5 + X^4 + 2X^3 + X^2 + X + 2 =$
$(X^4 + 2X^3 + X^2 - 4) \times (X - 1) + 3X^3 + 2X^2 + 5X - 2$
$X^4 + 2X^3 + X^2 - 4 =$
$(3X^3 + 2X^2 + 5X - 2) \times \frac{1}{9}(3X + 4) - \frac{14}{9}(X^2 + X + 2)$
$3X^3 + 2X^2 + 5X - 2 = (X^2 + X + 2) \times (3X - 1) + 0$
Thus $gcd(A, B) = X^2 + X + 2$.

Prim polynomials

Prim polynomials

Let $A, B \in \mathbb{K}[X]$. A and B are **prim** if $gcd(A, B) = 1$.

Bezout's theorem

Let $A, B \in \mathbb{K}[X]$ two polynomials with $A \neq 0$ or $B \neq 0$. We note $D = \mathrm{pgcd}(A, B)$. There exist $U, V \in \mathbb{K}[X]$ such that $AU + BV = D$.

corollary

Let $A, B \in \mathbb{K}[X]$ two polynomials. A and B are prim if and only if there exist two polynomials U and V such that $AU + BV = 1$.

corollary

Let $A, B, C \in \mathbb{K}[X]$ with $A \neq 0$ or $B \neq 0$. Si $C|A$ et $C|B$ then $C|\mathrm{pgcd}(A, B)$.

Roots and factorisation

Roots

let $P \in \mathbb{K}[X]$ and $\alpha \in \mathbb{K}$. We say that On dit que α is a **root** (or a **zero**) of P if $P(\alpha) = 0$. $P(\alpha) = 0 \iff X - \alpha$ divise P

Let $k \in \mathbb{N}^*$. α is a root of multiplicity k of P if $(X - \alpha)^k$ divide P and $(X - \alpha)^{k+1}$ don't divide P. We say that α is a root of order k.

These statement are equivalent :

(i) α is a root of multiplicity k of P.

(ii) There exist $Q \in \mathbb{K}[X]$ such that $P = (X - \alpha)^k Q$, with $Q(\alpha) \neq 0$.

(iii) $P(\alpha) = P'(\alpha) = \cdots = P^{(k-1)}(\alpha) = 0$ and $P^{(k)}(\alpha) \neq 0$.

Roots and factorisation

d'Alembert-Gauss's theorem

Any polynomial with complex coefficients of degree $n \geqslant 1$ has at least one root in $\mathbb{C}$. It has exactly n roots if we count each root with multiplicity.

Let $P(X) = aX^2 + bX + c$ a polynomial of degree 2 with real coefficients : $a, b, c \in \mathbb{R}$ and $a \neq 0$.

- If $\Delta = b^2 - 4ac > 0$ then P admit 2 distinct real roots $\frac{-b+\sqrt{\Delta}}{2a}$ and $\frac{-b-\sqrt{\Delta}}{2a}$.
- If $\Delta < 0$ then P admit 2 distinct complexes roots $\frac{-b+i\sqrt{|\Delta|}}{2a}$ et $\frac{-b-i\sqrt{|\Delta|}}{2a}$.
- If $\Delta = 0$ then P admit a double real root $\frac{-b}{2a}$.

With multiplicities, we always have exactly 2 roots.

Reducible and irreducible polynomials

Reducible polynomials

Let $P \in \mathbb{K}[X]$ a polynomial of degree $\geqslant 1$, we say that P is **irreducible** if for all $Q \in \mathbb{K}[X]$ divided P,then either $Q \in \mathbb{K}^*$, or there exists $\lambda \in \mathbb{K}^*$ such that $Q = \lambda P$.

- An irreducible polynomial P is therefore a non-constant polynomial whose only divisors of P are the constants or P itself (except for one multiplicative constant).
- The notion of an irreducible polynomial for the arithmetic of $\mathbb{K}[X]$ corresponds to the notion of prime number for $\mathbb{Z}$ arithmetic.
- Otherwise, P is said to be **reducible** ; then there exist polynomials A, B of $\mathbb{K}[X]$ such that $P = AB$, with $\deg A \geqslant 1$ and $\deg B \geqslant 1$.

Reducible and irreducible polynomials

Reducible and irreducible polynomials

- All polynomials of degree 1 are irreducible. Therefore there are infinitely many irreducible polynomials.
- $X^2 - 1 = (X-1)(X+1) \in \mathbb{R}[X]$ is reducible.
- $X^2 + 1 = (X-i)(X+i)$ is reducible on $\mathbb{C}[X]$ but irreducible on $\mathbb{R}[X]$.
- $X^2 - 2 = (X-\sqrt{2})(X+\sqrt{2})$ is reducible on $\mathbb{R}[X]$ but irreducible on $\mathbb{Q}[X]$.

Factorisation of polynomials

Factorisation's theorem

Any non-constant polynomial $A \in \mathbb{K}[X]$ can be written as a product of unitary irreducible irreducible polynomials :

$$A = \lambda P_1^{k_1} P_2^{k_2} \cdots P_r^{k_r}$$

where $\lambda \in \mathbb{K}^*$, $r \in \mathbb{N}^*, k_i \in \mathbb{N}^*$ and P_i are distinct irreducible polynomials. Moreover, this decomposition is unique to the nearest order of factors.

$$X^6 - 1 = (X^3+1)(X^3-1) = (X+1)(X^2-X+1)(X-1)(X^2+X+1)$$

$$X^6 - 1 = (X+1)(X-1)\left(X - \frac{1-i\sqrt{3}}{2}\right)\left(X - \frac{1+i\sqrt{3}}{2}\right)\left(X + \frac{1+i\sqrt{3}}{2}\right)\left(X + \frac{1-i\sqrt{3}}{2}\right)$$

Factorisation of polynomials

Factorisation of polynomials examples

- $P(X) = 2X^4(X-1)^3(X^2+1)^2(X^2+X+1)$ is already decomposed into irreducible factors in $\mathbb{R}[X]$. The decomposition in $\mathbb{C}[X]$ is $P(X) = 2X^4(X-1)^3(X-i)^2(X+i)^2(X-j)(X-j^2)$ where $j = e^{\frac{2i\pi}{3}} = \frac{-1+i\sqrt{3}}{2}$.
- Let $P(X) = X^4+1$.
  - On $\mathbb{C}$, $P(X) = (X^2+i)(X^2-i)$. So, $P(X) = \left(X - \frac{\sqrt{2}}{2}(1+i)\right)\left(X + \frac{\sqrt{2}}{2}(1+i)\right)\left(X - \frac{\sqrt{2}}{2}(1-i)\right)\left(X + \frac{\sqrt{2}}{2}(1-i)\right)$.
  - On $\mathbb{R}$. $P(X) = \left[\left(X - \frac{\sqrt{2}}{2}(1+i)\right)\left(X - \frac{\sqrt{2}}{2}(1-i)\right)\right]\left[\left(X + \frac{\sqrt{2}}{2}(1+i)\right)\left(X + \frac{\sqrt{2}}{2}(1-i\right.\right.$
    $\left[X^2+\sqrt{2}X+1\right]\left[X^2-\sqrt{2}X+1\right]$,

Rational fractions

Rational fractions

A **rational fraction** with coefficients in $\mathbb{K}$ is an expression of the form

$$F = \frac{P}{Q}$$

where $P, Q \in \mathbb{K}[X]$ are two polynomials and $Q \neq 0$.

Any rational fraction can be decomposed as a sum of elementary rational fractions which we call simple elements ». But the simple elements are different on $\mathbb{C}$ or on $\mathbb{R}$.

Root and pole of a rational fraction

Root and pole of a rational fraction

Let $F = \frac{P}{Q} \in \mathbb{K}[X]$ be an irreducible and non-zero rational fraction.

- The root of the fraction F is any root of the polynomial P in $\mathbb{K}[X]$.
- The scalar α of $\mathbb{K}$ is called the pole of F if it is a root of the polynomial Q in $\mathbb{K}[X]$.

$\rightsquigarrow$ The fraction $\frac{X^2(X+1)}{X-2}$ admit -1 as simple root and 0 as double root, and 2 as simple pole.

$\rightsquigarrow$ The fraction $\frac{(X-4)^3}{X^2+X+1}$ admit 4 as triple root and no pole on $\mathbb{R}$.

$\rightsquigarrow$ The fraction $\frac{2X+3}{X^2+X+1}$ don't admit a pole on $\mathbb{R}$, but admit two pole $\frac{-1-i\sqrt{3}}{2}$ and $\frac{-1+i\sqrt{3}}{2}$ on $\mathbb{C}$.

Decomposition into simple elements over $\mathbb{C}$

Theorem

Let P/Q be a rational fraction with $P, Q \in \mathbb{C}[X]$, $\mathrm{pgcd}(P,Q) = 1$ and $Q = (X-\alpha_1)^{k_1} \cdots (X-\alpha_r)^{k_r}$. Then there is one and only one notation :

$$\begin{aligned}\frac{P}{Q} = \quad E & + \frac{a_{1,1}}{(X-\alpha_1)^{k_1}} + \frac{a_{1,2}}{(X-\alpha_1)^{k_1-1}} + \cdots + \frac{a_{1,k_1}}{(X-\alpha_1)} \\ & + \frac{a_{2,1}}{(X-\alpha_2)^{k_2}} + \cdots + \frac{a_{2,k_2}}{(X-\alpha_2)} \\ & + \cdots\end{aligned}$$

The polynomial E is called the **polynomial part** (or **whole part**). The terms $\frac{a}{(X-\alpha)^i}$ are the **simple elements** on $\mathbb{C}$.

Decomposition example

$$\frac{P}{Q} = \frac{X^5 - 2X^3 + 4X^2 - 8X + 11}{X^3 - 3X + 2}$$

- **Step 1 : polynomial part.**
  Using the Euclidean division of P by Q :
  $P(X) = (X^2 + 1)Q(X) + 2X^2 - 5X + 9$. So, $E(X) = X^2 + 1$ and $\frac{P(X)}{Q(X)} = X^2 + 1 + \frac{2X^2 - 5X + 9}{Q(X)}$. Where
  $\deg P_1 = 2X^2 - 5X + 9 < \deg Q$
- **Step 2 : factorisation of Q.**
  The pole of the fraction is when $Q = 0$, so the roots are a pour $+1$ (root double) and -2 (simple root), the factorisation is $Q(X) = (X - 1)^2(X + 2)$.

Decomposition example

$$\frac{P}{Q} = \frac{X^5 - 2X^3 + 4X^2 - 8X + 11}{X^3 - 3X + 2}$$

⇝ **Step 3 : Theoretical decomposition on to simple elements.**

Using the decomposition theorem :

$$\frac{P(X)}{Q(X)} = E(X) + \frac{a}{(X-1)^2} + \frac{b}{X-1} + \frac{c}{X+2}$$

Where $E(X) = X^2 + 1$. We have to determine a, b, c.

Decomposition example

$$\frac{P}{Q} = \frac{X^5 - 2X^3 + 4X^2 - 8X + 11}{X^3 - 3X + 2}$$

⇝ **Step 4 (V_1) : determinate coefficients.** We write $\frac{a}{(X-1)^2} + \frac{b}{X-1} + \frac{c}{X+2}$ with same denominator and we identify with $\frac{P_1 = 2X^2-5X+9}{Q(X)}$, so $\frac{a}{(X-1)^2} + \frac{b}{X-1} + \frac{c}{X+2} = \frac{(b+c)X^2+(a+b-2c)X+2a-2b+c}{(X-1)^2(X+2)} = \frac{2X^2-5X+9}{(X-1)^2(X+2)}$. We deduce that $b+c=2$, $a+b-2c=-5$ and $2a-2b+c=9$. thus $a=2$, $b=-1$, $c=3$. and

$$\frac{P}{Q} = X^2 + 1 + \frac{2}{(X-1)^2} + \frac{-1}{X-1} + \frac{3}{X+2}.$$

This method often takes the longest.

Decomposition example

$$\frac{P}{Q} = \frac{X^5 - 2X^3 + 4X^2 - 8X + 11}{X^3 - 3X + 2}$$

- **Step 4 (V_2) : determinate coefficients.**
  $\frac{P_1(X)}{Q(X)} = \frac{2X^2-5X+9}{(X-1)^2(X+2)} = \frac{a}{(X-1)^2} + \frac{b}{X-1} + \frac{c}{X+2}$ To get a we multiply the fraction $\frac{P_1}{Q}$ by $(X-1)^2$ and we evaluate on $x=1$.
  $F_1(X) = (X-1)^2 \frac{P_1(X)}{Q(X)} = a + b(X-1) + c\frac{(X-1)^2}{X+2}$, $F_1(1) = a$
  And $F_1(X) = \frac{2X^2-5X+9}{X+2}$ so, $F_1(1) = 2$. We deuce that $a = 2$.
  The same process is used to determine c :
- $F_2(-2) = c$ and $F_2(-2) = 3$, $c = 3$
- For b we can use : $\frac{P_1(0)}{Q(0)} = a - b + \frac{c}{2}$ thus $\frac{9}{2} = a - b + \frac{c}{2}$. So $b = a + \frac{c}{2} - \frac{9}{2} = -1$.

Decomposition example

$\frac{P(X)}{Q(X)} = \frac{3X^4+5X^3+11X^2+5X+3}{(X^2+X+1)^2(X-1)}$

- Since $\deg P < \deg Q$ then $E(X) = 0$.
- $X^2 + X + 1$ is irreducible on $\mathbb{R}$,

$$\frac{P(X)}{Q(X)} = \frac{aX+b}{(X^2+X+1)^2} + \frac{cX+d}{X^2+X+1} + \frac{e}{X-1}.$$

- After calculating coefficients, we get :

$$\frac{P(X)}{Q(X)} = \frac{2X+1}{(X^2+X+1)^2} + \frac{-1}{X^2+X+1} + \frac{3}{X-1}.$$

Decomposition example

$$\int_1^2 \frac{3x^2+2x+1}{x(x^2+x+1)}dx$$

$\rightsquigarrow$ $\frac{P(X)}{Q(X)} = \frac{3x^2+2x+1}{x(x^2+x+1)}$. $\deg P < \deg Q$, so $E(X) = 0$ and (x^2+x+1) is irreducible on $\mathbb{R}$, thus $\frac{3x^2+2x+1}{x(x^2+x+1)} = \frac{a}{x} + \frac{bx+c}{x^2+x+1}$

$\rightsquigarrow$ $3X^2+2X+1 = (a+b)X^2+(a+c)X+a$, $a = 1, b = 2$ and $c = 1$.

$\rightsquigarrow$ $\displaystyle\int_1^2 \frac{3x^2+2x+1}{x(x^2+x+1)}dx = \int_1^2 \frac{dx}{x} + \int_1^2 \frac{2x+1}{x^2+x+1}dx = [ln(x)]_1^2 + [ln(x^2+x+1)]_1^2$

Decomposition example on $\mathbb{C}$

$\frac{P(X)}{Q(X)} = \frac{3X+1}{X^2+1}$

- $\rightsquigarrow$ $\deg P < \deg Q$, so $E(X) = 0$ and there are two poles $(-i, i)$ on $\mathbb{C}$, $(X^2+1) = (X-i)(X+i)$. $\frac{3X+1}{X^2+1} = \frac{a}{X-i} + \frac{b}{X+i}$.
- $\rightsquigarrow$ We multiply by $(X-i)$ and evaluate on $X = i$, $a = \frac{3}{2} - \frac{1}{2}i$;
- $\rightsquigarrow$ We multiply by $(X+i)$ and evaluate on $X = -i$, $b = \frac{3}{2} + \frac{1}{2}i$;
- $\rightsquigarrow$ $\frac{3X+1}{X^2+1} = \frac{\frac{3}{2}-\frac{1}{2}i}{X-i} + \frac{\frac{3}{2}+\frac{1}{2}i}{X+i}$

Decomposition example

Consider in $\mathbb{R}[X]$ the polynomial
$P = X^5 - 3X^4 + 5X^3 - 7X^2 + 6X - 2.$

1. Verify that 1 is a multiple root of P, give his multiplicity and factorise P into irreducible polynomials on $\mathbb{R}[X]$ and on $\mathbb{C}[X]$.
2. Decompose into simple elements the followings rational fraction : $F = \dfrac{X^2+3}{X^5 - 3X^4 + 5X^3 - 7X^2 + 6X - 2}$

$P = X^5 - 3X^4 + 5X^3 - 7X^2 + 6X - 2.$

$\rightsquigarrow$ $P(1) = P'(1) = P''(1) = 0$ and $P^{(3)}(1) \neq 0$. 1 is a multiple root of multiplicity 3 and P is divisible by $(X-1)^3$, using the Euclidean division we find
$P = (X-1)^3(X^2+2) = (X-1)^3(X - i\sqrt{2})(X + i\sqrt{2})$ witch are the factorisation on $\mathbb{R}$ and $\mathbb{C}$

$\rightsquigarrow$

Decomposition example

Consider in $\mathbb{R}[X]$ the polynomial
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$F = \frac{X^2+3}{X^5-3X^4+5X^3-7X^2+6X-2}$

$\rightsquigarrow$ $F = \frac{X^2+3}{(X-1)^3(X^2+2)} = \frac{X^2+3}{(X-1)^3(X-i\sqrt{2})(X+i\sqrt{2})}$

$\rightsquigarrow$ $F = \frac{a}{(X-1)^3} + \frac{b}{(X-1)^2} + \frac{c}{(X-1)} + \frac{dX+e}{(X^2+2)}$ or

$F = \frac{a}{(X-1)^3} + \frac{b}{(X-1)^2} + \frac{c}{(X-1)} + \frac{d}{(X-i\sqrt{2})} + \frac{e}{(X+i\sqrt{2})}$

Decomposition example

Consider $F = \frac{X^2+3}{(X-1)^3(X^2+2)}$ and take $Y = X - 1$

- $\rightsquigarrow$ $F(Y) = \frac{Y^2+2Y+4}{(Y^3(Y^2+2Y+3)}$
- $\rightsquigarrow$ The successive increasing Euclidean division give $4+2Y+Y^2 = (3+2Y+Y^2)(\frac{4}{3} - \frac{2}{9}Y + \frac{1}{27}Y^2) + \frac{1}{27}Y^3(4-Y)$

$$
\begin{array}{ccc|c}
4 & +2Y & +Y^2 & 3 \quad +2Y \quad +Y^2 \\
\hline
4 & \frac{8}{3}Y & \frac{4}{3}Y^2 & \frac{4}{3} \quad -\frac{2}{9}Y \quad +\frac{1}{27}Y^2 \\
\hline
 & \ddots & & \\
 & & \ddots & \\
\hline
 & \frac{4}{27}Y^3 & -\frac{1}{27}Y^4 &
\end{array}
$$

Decomposition example

Consider $F=\frac{X^2+3}{(X-1)^3(X^2+2)}$ and take $Y=X-1$

$\rightsquigarrow$ $F(Y)=\frac{Y^2+2Y+4}{(Y^3(Y^2+2Y+3))}=\frac{(3+2Y+Y^2)(\frac{4}{3}-\frac{2}{9}Y+\frac{1}{27}Y^2)+\frac{1}{27}Y^3(4-Y)}{(Y^3(Y^2+2Y+3)}$

$\rightsquigarrow$ $F(Y)=\frac{(\frac{4}{3}-\frac{2}{9}Y+\frac{1}{27}Y^2)}{Y^3}+\frac{\frac{1}{27}(4-Y)}{(Y^2+2Y+3)}=$
$\frac{\frac{4}{3}}{Y^3}+\frac{-\frac{2}{9}}{Y^2}+\frac{\frac{1}{27}}{Y}+\frac{\frac{1}{27}(4-Y)}{(Y^2+2Y+3)}$

$\rightsquigarrow$ $F(X)=\frac{X^2+3}{(X-1)^3(X^2+2)}=\frac{\frac{4}{3}}{(X-1)^3}+\frac{-\frac{2}{9}}{(X-1)^2}+\frac{\frac{1}{27}}{(X-1)}+\frac{\frac{1}{27}(-X+5)}{(X^2+2)}$

Decomposition example

1. Decompose into simple elements the followings rational fraction : $F = \dfrac{X^3 - X^2 + 2X - 3}{(X^2 + X + 1)^2}$
2. Let P the polynomial of $\mathbb{R}[X]$ defined by $P = X^6 + 4X^5 + 8X^4 + 10X^3 + 8X^2 + 4X + 1$. Without using the factorisation of P given below, check that -1 is a double root (in $\mathbb{R}$) of P and that j and $\bar{j}$ are two double roots (in $\mathbb{C}$) of P. Deduce that $P = (X+1)^2(X^2 + X + 1)^2$.
3. Consider the following rational faction of $R(X)$: $F = \dfrac{X^2 - 3X - 2}{X^6 + 4X^5 + 8X^4 + 10X^3 + 8X^2 + 4X + 1}$ Check that $F = \dfrac{X^2 - 3X - 2}{((X-\alpha)^2 C^2)}$ with $\alpha \in \mathbb{R}$ and $C \in \mathbb{R}[X]$. Using division by increasing powers, calculate $\lambda_1 \in \mathbb{R}, \lambda_2 \in \mathbb{R}$ and $R \in \mathbb{R}[X]$ such that $F = \dfrac{\lambda_1}{(X-\alpha)} + \dfrac{\lambda_2}{(X-\alpha)^2} + \dfrac{R}{C^2}$ Complete the

Decomposition example (1)

$$F = \frac{X^3 - X^2 + 2X - 3}{(X^2 + X + 1)^2}$$

- $\rightsquigarrow$ The euclidean division of $X^3 - X^2 + 2X - 3$ by $X^2 + X + 1$ give : $X^3 - X^2 + 2X - 3 = (X^2 + X + 1)(X - 2) + (3X - 1)$
- $\rightsquigarrow$ So,

$$F = \frac{X^3 - X^2 + 2X - 3}{(X^2 + X + 1)^2} = \frac{X - 2}{X^2 + X + 1} + \frac{3X - 1}{(X^2 + X + 1)^2}$$

Decomposition example (2)

$P = X^6 + 4X^5 + 8X^4 + 10X^3 + 8X^2 + 4X + 1.$

- $P(-1) = 0$, $P'(-1) = 0$ and $P''(-1) \neq 0$. -1 is a double root of P.
- $P(j) = 0$, $P'(j) = 0$ and $P''(-1) \neq 0$. (same thing for $\bar{j}$) j and $\bar{j}$ are a double root of P.

$$(X - j)^2(X - \bar{j})^2 = (X^2 + X + 1)^2$$

- Since P is unitary polynomial

$$P = X^6+4X^5+8X^4+10X^3+8X^2+4X+1 = (X+1)^2(X^2+X+1)^2$$

Decomposition example (3)

Consider $F = \frac{X^2-3X-2}{(X+1)^2(X^2+X+1)^2}$ and take $Y = X+1$

- $\rightsquigarrow$ $F(Y) = \frac{Y^2-5Y+2}{Y^2(Y^4-2Y^3+3Y^2-2Y+1)}$
- $\rightsquigarrow$ The successive increasing Euclidean division give
  $2-5Y+Y^2 =$
  $(1-2Y+3Y^2-2Y^3+Y^4)(2-Y)-7Y^2+7Y^3-4Y^4+Y^5$
- $\rightsquigarrow$ $F(Y) = \frac{Y^2-5Y+2}{Y^2(Y^4-2Y^3+3Y^2-2Y+1)} =$
  $\frac{-1}{Y} + \frac{2}{Y^2} + \frac{-7+7Y-4Y^2+Y^3}{Y^4-2Y^3+3Y^2-2Y+1}$